BOSTON UNIVERSITY

ALGEBRAIC TRANSFORMATIONS OF INFINITE REAL SERIES

Thesis

Submitted by

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(A.B., Colby College, 1922)

In Partial Fulfillment of the Requirements

for the Degree of

Master of Arts

1938

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ALGEBRAIC TRANSFORMATIONS OF INFINITE SERIES

Part I. Definitions and Theorems assumed

A. Definitions:

1. Sequence
2. Series
3. Partial sum of a series
4. Convergent series
   I. Properly divergent
   II. Oscillating
5. Alternating Series
   - Absolute and conditional convergence
6. Monotonic Sequences

B. Theorems Assumed:

1. Every bounded infinite sequence has at least one limiting point.
2. A bounded monotonic sequence has at least one limiting point.
3. The general convergence theorem.
4. A series of positive terms is either steadily divergent or is convergent.
5. An alternating series is convergent if:

   (1) $u_{n+1} \leqq u_n$ (2) $\lim\limits_{n\to\infty} u_n = 0$

6. The general test for alternating series: by dividing into a positive and negative series and testing each of these.
7. Convergence and divergence of the geometric series.
8. Convergence and divergence of the hyperharmonic series.
9. Comparison tests for convergence and divergence.
10. D'Alembert's Criterion for convergence and divergence.
11. Raabe's Criterion for convergence and divergence.

PART I

A. Definitions

1. If an infinite aggregate of real or complex elements is such that we can establish a 1 to 1 correspondence between the elements of the aggregate and the set of positive integers 1, 2, 3, ----, n, -------we call the aggregate an infinite sequence. An infinite sequence is considered as defined, when some law or rule is given by which we can determine the n^{th} term when an index n is assigned. A sequence is usually given by a formula for its n^{th} term or by means of the first few terms, from which the n^{th} term may be found or by a combination of both methods.

Ex. (1) In the sequence $1/2, -2/3, +3/4, -4/5$, ---the n^{th} term is $u_n = (-1)^{n+1} \dfrac{n}{n+1}$

Ex. (2) If the n^{th} term is, $u_n = \dfrac{n^2}{n+1}$ the first few terms are 1/2, 4/3, 9/4, 16/5, 25/6, -----

2. If u_1, u_2, u_3, ----, u_n---- is an infinite sequence of real or complex terms, the symbol

$u_1+u_2+u_3+$ ------ $+u_n+$ ---- is called an infinite series.

As with the sequence the formula for the n^{th} term may be given, from which we may write any term, or the first few terms may be given and from these, the formula for the n^{th} term may be derived, or we may use a combination of both methods.

Ex. (1) Given the series whose first few terms are

$$\frac{4}{1\cdot 3}+\frac{5}{2\cdot 4}+\frac{6}{3\cdot 5}+ \text{------ the } n^{th} \text{ term} = \frac{n+3}{n(n+2)}$$

Ex. (2) If the n^{th} term is $u_n = \frac{2n+1}{n^2+1}$ the series has for its first few terms, for n=1, 2, 3, ----, 3/2 + 5/5 + 7/10 + ---. For the series $u_1+u_2+u_3+$ ---- $+ u_n +$ ---- we often use the abbreviated notation $\sum u_n$ which is read summation of u_n, or $\sum_{n=1}^{\infty} u_n$ which is read the summation of u_n from n=1 to n= ∞ .

3. If we put

$$s_n = u_1+u_2+u_3+ \text{----} + u_n = \sum_{i=1}^{n} u_i$$ then s_n is the sum of the first n terms of the series and is called a partial sum of the series $\sum u_n$.

4. In general, series are divided into two classes, convergent and divergent. Divergent series may be further divided into two classes, properly divergent and oscillating. An infinite series is defined to be convergent if the sum of the first n terms approaches a definite limit when n becomes infinite. For example, let u_1, u_2, u_3, ---, u_n be a series of terms whose order of succession follows some general law.

Let

$$s_n = u_1+u_2+u_3+ \text{----} + u_n$$

then if the limit s_n = A (some definite limit) the series is then said to be convergent.

I. Divergent series are by definition non-convergent. In the case of the properly divergent series s_n increases without limit, consequently no limit for s_n exists. II. In an oscillating series s_n is bounded but $\lim_{n \to \infty} s_n$ does not exist.

5. The alternating series, a series whose terms are alternately positive and negative, leads us to make a new classification of series, those which converge absolutely and those which are conditionally convergent. If the series of absolute values formed from a given series converges, then the given series converges absolutely. If the series of absolute values is divergent then the given series may be conditionally convergent.

6. A sequence is said to be monotonic non-decreasing if for every n

$$u_n \leqq u_{n+1}$$

It is non-increasing if for every n

$$u_n \geqq u_{n+1}$$

A sequence is bounded if

$$o \leqq |u_n| \leqq b$$

A series of real terms is bounded if as above

$$o \leqq |s_n| \leqq b$$

Particularly if

$$a \leqq |s_n| \leqq b$$

B. Theorems Whose Proofs are Assumed.

1. Every bounded infinite sequence has at least one limiting point.

2. A bounded monotonic sequence has one and only one limiting point.

3. The general convergence theorem. A necessary and sufficient condition for convergence of a series $\sum u_n$ is that for every $\epsilon > 0$ however small, an index N_ϵ can be found such that we have

$$|s_{N+k} - s_N| < \epsilon \qquad k = 1, 2, 3, \text{-----}$$

4. A series of positive terms $\sum u_n$ is convergent if s_n is less than some fixed value for every n. If the series is not convergent then it is properly divergent, that is, for any positive number G however large there exists an N_G such that $S_N > G$ for every $n > N_G$.

5. An alternating series is convergent if each term is numerically less than the preceding, and $\lim u_n = 0$.

6. If $\sum u_n$ is made up of positive terms (a_n) and negative terms $(-b_n)$ and if $\sum a_n$ and $\sum - b_n$ are both convergent, then $\sum u_n$ is absolutely convergent: If both $\sum a_n$ and $\sum -b_n$ are divergent $\sum u_n$ is at most conditionally convergent, if convergent at all, and conversely if $\sum u_n$ is conditionally convergent then $\sum a_n$ and $\sum -b_n$ must be divergent: and if $\sum u_n$ is absolutely convergent then $\sum a_n$ and $\sum -b_n$ are both convergent. If one of the series $\sum a_n$, $\sum -b_n$ is convergent

and one divergent then the $\sum u_n$ is divergent.

7. The series $1 + r + r^2 + r^3 + ---- + r^n + ---- = \sum_{n=0}^{\infty} r^n$

in which the ratio of any term to its preceding one is a constant r is called a geometric series and its convergence and divergence may be summarized thus:

I. Convergent if $|r| < 1$; its sum is $\frac{1}{1-r}$

II. Properly divergent if $|r| \geqq 1$

III. Oscillating if $r \leqq -1$

8. The series

$$1 + \frac{1}{2^k} + \frac{1}{3^k} + ----- + \frac{1}{n^k} + -------$$

is called the hyperharmonic series. It includes the harmonic series, which is the special case where $k = 1$. The hyperharmonic series can be shown to be properly divergent for $k \leqq 1$ and convergent for $k > 1$.

Tests for convergence:

9. Direct Comparison. (A) Let

$$u_1 + u_2 + u_3 + ---- + u_n + -------$$

be a series of positive terms to be tested. If a series of positive terms already known to be convergent,

$$a_1 + a_2 + a_3 + ---- + a_n + -------$$

can be found such that $u_n \leqq a_n$ then the given series is convergent, and its value does not exceed that of the known series.

(B) Let

$$u_1 + u_2 + u_3 + \text{--------} + u_n + \text{----}$$

be a series of positive terms to be tested for divergence. If a divergent series

$$a_1 + a_2 + a_3 + \text{-----} + a_n + \text{----}$$

can be found such that $u_n \geqq a_n$ then the given series is divergent.

10. D'Alembert's Ratio Test. If the Lim $\left|\frac{u_{n+1}}{u_n}\right|$ exists = T then $\sum u_n$ is convergent when $T < 1$, divergent when $T > 1$, and no test is given when T = 1.

11. Raabe's Test. If the Lim $n\left(\frac{u_n}{u_{n+1}} - 1\right)$ exists = R then $\sum u_n$ is convergent if $R > 1$, and is divergent if $R < 1$ and no test is given if R = 1.

Part II

Since an infinite series is the limit of a sum and not a sum, processes applicable to a sum need not be applicable to a series. If they are to be applied, this fact will require proof. We shall now see under what restriction, if any, the associative and commutative laws of algebra hold in infinite series - and see under what condition series may be added, multiplied or divided.

Theorem 1: The terms of a convergent series can be united into groups in any manner without affecting the sum of the series, provided the order of terms is not changed.[1]

Let

$$s_n = u_1 + u_2 + u_3 + \text{-------------} + u_n$$

and denote the sum of the first m groups of terms by

$$S'_m = U_1 + U_2 + U_3 \text{ ------------} + U_m$$

However large n may be, m may be taken sufficiently large so that S'_m will contain at least all of the terms of s_n. If S'_m contains $n+p$ terms of the given series $\sum u_n$ we have

$$S'_m - s_n = u_{n+1} + u_{n+2} + \text{-------} + u_{n+p} \qquad p = 1, 2, 3, \text{---}$$

Denoting the second member of this equation by $\sum_{n+1}^{n+p} u_n$ we have by

1. Townsend, Edgar J., Functions of Real Variable, p. 338.

the general convergence theorem[1] for any arbitrarily chosen ϵ $\left|\sum_{n+1}^{n+p} u_n\right| < \epsilon$

for all values of n sufficiently large. As n becomes infinite, the value of m also increases without limit, hence

$$\left|S'_m - s_n\right| < \epsilon$$

or

$$\lim_{m\to\infty} S'_m = \lim_{n\to\infty} s_n$$

This theorem shows that we can always insert parentheses at pleasure in a series, but parentheses cannot be removed from a convergent series except under proper conditions. Take a simple series $(1-1) + (1-1) + (1-1) + ---- = \sum (1-1) = 0$ which is convergent but remove the parentheses and we have $1-1+1-1+$ ------- a divergent series (oscillating), which cannot be put equal to zero. By still another grouping we might obtain the series $1-(1-1)-(1-1)$ ------ $= 1-0-0-0$ and $1 \neq 0$ the value of the original series.

Theorem 2: If a series involving parentheses converges absolutely after the parentheses have been removed, then the removal of the parentheses does not affect the sum of the series.[2]

Let

$$S'_m = U_1 + U_2 + U_3 + ----- + U_m$$

where each U represents the content of a parenthesis taken as one term and let

$$s_n = u_1 + u_2 + u_3 + ------- + u_n$$

1. Theorem 3, p. 4
2. Townsend, Edgar J., <u>Functions of Real Variable</u>, p. 339.

be the series after the parentheses have been removed. The value of m may certainly be taken sufficiently large to contain at least the n terms of s_n. It follows that

$$|S'_m - s_n| \leqq u_{n+1} + u_{n+2} + \text{-------} + u_{n+p}$$

where n+p denotes the number of terms of the series which are included in the first m parentheses. From the absolute convergence of series after the parentheses have been removed, we know that the right-hand member of this inequality may be made as small as we please by taking n sufficiently large, say $n > n'$. We have then by the general convergence theorem, for an arbitrarily small ϵ remembering that as n increases indefinitely m does also

$$|S'_m - s_n| < \epsilon \,, \quad n > n', \quad m > m'$$

hence

$$\lim_{m\to\infty} S'_m = \lim_{n\to\infty} s_n$$

Theorem 3: In an absolutely convergent series, the terms can be rearranged at will without altering the value of the series.[1]

Case 1: First suppose all the terms to be positive (or all negative). Let

$$s_n = u_1 + u_2 + u_3 + \text{-------} + u_n \qquad \lim_{n\to\infty} s_n = U$$ be the

series and let

$$S'_{n'} = u'_1 + u'_2 + \text{---------} + u'_{n'}.$$

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1. Osgood, William F., <u>Introduction to Infinite Series</u>, p. 44.

be the series after rearrangement.

Then $S'_{n'}$ approaches the limit U, when $n' \to \infty$, for $S'_{n'}$ always increases as n' increases (terms are all positive). No matter how large n' be taken (and held fast) n can be taken so large that s_n will contain all the terms of $S'_{n'}$ and more too. Therefore, $S'_{n'} \leq s_n \leq U$, or no matter how large n' be taken $S'_{n'} \leq U$, or $S'_{n'}$ approaches a limit $U' \leqq U$. Since $S'_{n'}$ is less than or equal to s_n, the sequence is bounded by U and every bounded sequence has at least one limit. By turning things about let the u series be the rearrangement of the u' series, and we have $U \leq U'$. Therefore U = U' since U cannot be both greater and smaller than U'.

Case 2: If the series

$$u_1+u_2+u_3+ \text{------} +u_n+ \text{------}$$

contains both positive and negative terms let

$$v_1+v_2+v_3+ \text{------} +v_n+ \text{----}$$

be the series of positive terms and

$$-w_1-w_2-w_3- \text{--------} -w_n- \text{-----}$$

be the series of negative terms in the order in which they occur.

Let

$$S_n = u_1+u_2+u_3+ \text{------} +u_n$$

$$s_m = v_1+v_2+v_3+ \text{------} +v_m$$

$$-t_p = -w_1-w_2-w_3- \text{-------} -w_p$$

Then for any value of n, S_n can be written in the form

$$S_n = s_m-t_p$$

Where m denotes the number of positive terms in S_n, s_m their sum, p

denotes the number of negative terms and t_p their sum.

Since the u series is absolutely convergent, the v series and w series are convergent. (Part 1 B 6)

$$\lim_{m \to \infty} s_m = V \qquad \lim_{p \to \infty} -t_p = -W$$

$$\therefore U = V - W \qquad \left(U = \lim_{n \to \infty} \sum u_n\right)$$

Let

$u'_1 + u'_2 + u'_3 + \text{-------} + u'_n + \text{------}$ be the series after the rearrangement and as above

$$S'_{n'} = s'_{m'} - t'_{p'}$$

$$U' = V' - W'$$

But $V = V'$ and $W = W'$ since v series and w series are convergent they may be arranged in any order.[1]

$\therefore$ $U = U'$ and the series after rearrangement approaches the same limit as the original series.

The rearrangement of terms aids in finding the sum of the following series

$$\frac{1}{2} + \frac{1}{2^3} - \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^7} - \frac{1}{2^4} + \frac{1}{2^9} + \frac{1}{2^{11}} - \frac{1}{2^6} + \text{-------}$$

In this series every third term is negative and contains an even power of 2. The terms containing the odd powers of 2 are positive and arranged in ascending order of those powers. If we take the series of

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1. By page 4, theorem 6, the v series and the w are convergent, and since all the terms in each have the same sign, they are absolutely convergent, and may be arranged in any order by Case I.

positive terms

$$\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+ \text{--------}$$

and the series of negative terms

$$-\frac{1}{2^2}-\frac{1}{2^4}-\frac{1}{2^6}- \text{--------}$$

we find that each is a geometric series with ratio $\frac{1}{4}$ and is absolutely convergent. Since both of these series are convergent the original series is absolutely convergent. Hence by the foregoing theorem, the terms may be arranged in any order without affecting the sum. We may then write the series as follows:

$$\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^5}-\frac{1}{2^6}+\frac{1}{2^7}- \text{----------}$$

$$= \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}-\frac{1}{2^5}+\frac{1}{2^6}- \text{---}\right)$$

$$= \frac{1}{2}\left[\left(1+\frac{1}{2^2}+\frac{1}{2^4}+ \text{---}\right) - \left(\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+ \text{---}\right)\right]$$

The terms in each parentheses are a geometric progression whose sum = $\frac{1}{1-r}$

$$= \frac{1}{2}\left\{\frac{1}{1-\frac{1}{4}} - \frac{1}{2}\cdot\frac{1}{1-\frac{1}{4}}\right\} = \frac{1}{2}\left\{\frac{4}{3}-\frac{2}{3}\right\} = \frac{1}{3}$$

Theorem 4: The terms of a conditionally convergent series can always be arranged so that (I) the new series is convergent with any arbitrarily given number as a sum, or (II) so that the new series is divergent, or (III) so that the new series may oscillate between arbitrarily assigned bounds and each of these rearrangements may be made in infinitely many ways.[1]

(I) Let $u_1+u_2+u_3+$ ----- $+u_n+$ --- be the terms of a conditionally convergent series and let

1. Riemann's Theorem.

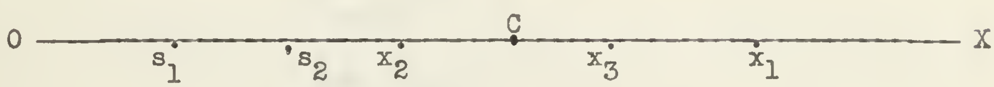
$$v_1 + v_2 + v_3 + \text{----} + v_n + \text{---}$$

be the series of positive terms and

$$-w_1 - w_2 - w_3 - \text{-----} - w_n - \text{---}$$

be the series of negative terms as they occur in the u series. If the v series and w series both divergent then the u series is at most only conditionally convergent, or can be made to converge to any arbitrary constant.

Let C be any arbitrary point on the line OX:



Represent the partial sums s_1, s_2, s_3,--- of the sequence in the usual way by points on the line OX. Since the u series is conditionally convergent, therefore the v series is divergent[1], some of these points must lie on the opposite side of C from the origin. Let x_1 be the first of such points, and let the corresponding sum in the sequence be s_k. To this result add a sufficient number of terms in the series $-w_1-w_2-w_3-$ ---- so that ultimately we get x_2 such that C lies between x_2 and x_1. Then we add enough terms from the v series to put x_3 on the other side of C from x_2 then add enough terms from the w series to get x_4 on the origin side of C and so on.

There will be values V and W such that all the x points must lie in the interval,

$$C - W \leq x \leq C + V$$

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1. Part I, B, theorem 6.

since $\lim_{n \to \infty} v_n = 0$ and $\lim_{n \to \infty} - w_n = 0$ is a necessary condition for convergence. But now the set of x points is bounded. Hence they have at least one limiting point X. But from the method of construction

$$|x_n - C| < v_k \text{ or } w_\ell$$

where v_k and w_ℓ are the last terms necessary to be added to each sum formed to make it large enough to cross to the other side of point C.

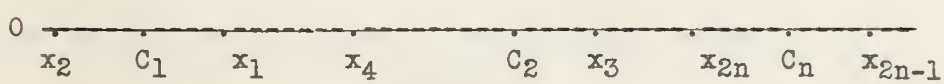
Since the given series is convergent

$$\lim_{k \to \infty} v_k = 0$$

$$\lim_{\ell \to \infty} w_\ell = 0$$

Hence $$\lim_{n \to \infty} x_n = C$$

(II) To effect divergence we have only to take a sequence of constants with $\lim_{n \to \infty} C_n = \infty$ $\qquad C_1 < C_2 < \text{---------} < C_n$



Since the v series is divergent add enough terms so that we get

$$C_1 < x_1$$

then add enough terms of the w series to get

$$x_2 < C_1$$

Now add more terms from the v series to get

$$C_2 < x_3$$

then add more terms from the w series to get

$$x_4 < C_3$$

and so on

$$C_n < x_{2n-1}$$

$$x_{2n} < C_n$$

and we then have divergence.

(III) To get oscillation between two arbitrarily assigned bounds, take the constants C_1 and C_2, $C_1 < C_2$.

Since the series converges take enough terms of the v series to get $x_1 < C_2$; should the first few terms of the v series be large making $x_1 > C_2$, add a few terms of the w series to get $x_1 < C_2$, or should the first few terms of the w series be large and make $x_1 < C_1$ take a few more terms of the v series which is divergent to get $x_1 > C_1$ so that we have

$$C_1 < x_1 < C_2$$

and now add enough terms of the w series to get

$$x_2 > C_1$$

then add more terms from the v series until

$$x_3 < C_2$$

now add more terms from the w series until we get

$$x_4 > C_1$$

and so on-----

We then have oscillation between C_1 and C_2.

We see then, from the above discussion, that the commutative law of algebra holds unrestrictedly for absolutely convergent series.

The rearrangement of terms should be made according to some law, but it is not necessary to know what that law is.

Example: to show that a change in the order of terms of a conditionally convergent series produces a change in the value of the series:

Consider the conditionally convergent series

$$V = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \text{------} + \left(\frac{1}{2m+1} - \frac{1}{2m+2}\right) \quad m = 0, 1, 2, 3, \text{----}$$

The value of this series may be written

$$\operatorname{Lim} \sum_{m=0}^{\infty} \left(\frac{1}{2m+1} - \frac{1}{2m+2}\right)$$

If we write the terms in another order, say putting two negative terms after each positive term

$$V' = \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{2m+1} - \frac{1}{4m+2} - \frac{1}{4m+4}\right)$$

Apply Raabe's test to this series

$$u_m = \frac{1}{2m+1} - \frac{1}{4m+2} - \frac{1}{4m+4} = \frac{1}{4(2m+1)(m+1)}$$

$$u_{m+1} = \frac{1}{2m+3} - \frac{1}{4m+6} - \frac{1}{4m+8} = \frac{1}{4(2m+3)(m+2)}$$

$$\operatorname*{Lim}_{m \to \infty} m \left\{ \frac{\dfrac{1}{4(2m+1)(m+1)}}{\dfrac{1}{4(2m+3)(m+2)}} - 1 \right\} = \operatorname*{Lim}_{m \to \infty} m \left\{ \frac{2m^2 + 7m + 6}{2m^2 + 3m + 1} - 1 \right\} =$$

$$\lim_{m\to\infty} m \left(\frac{2m^2 + 7m + 6 - 2m^2 - 3m - 1}{2m^2 + 3m + 1}\right) = \lim_{m\to\infty}\left(\frac{4m^2 + 5m}{2m^2 + 3m + 1}\right) =$$

$$\lim_{m\to\infty}\left(\frac{4 + \frac{5}{m}}{2 + \frac{3}{m} + \frac{1}{m^2}}\right) = 2.$$ Hence $R > 1$ and the series is convergent.

The value of V' may then be written

$$\lim \sum_{m=0}^{\infty}\left(\frac{1}{2m+1} - \frac{1}{4m+2} - \frac{1}{4m+4}\right)$$

By comparing the values of the general terms in V and V'

$$(V') \qquad \frac{1}{2m+1} - \frac{1}{4m+2} - \frac{1}{4m+4} = \frac{1}{4(2m+1)(m+1)}$$

$$(V) \qquad \frac{1}{2m+1} - \frac{1}{2(m+1)} = \frac{1}{2(2m+1)(m+1)}$$

or

$$\frac{1}{4(2m+1)(m+1)} = \frac{1}{2}\left(\frac{1}{2(2m+1)(m+1)}\right)$$

We find that

$$\lim \sum_{m=0}^{\infty} V = \frac{1}{2} \lim \sum_{m=0}^{\infty} V'$$

The summation of the new arrangement is equal to $\frac{1}{2}$ the summation of the old. This is only one of an infinite number of arrangements.

Addition of Series

Theorem 5:

If $U = u_1 + u_2 + u_3 + \text{--------}$

and $V = v_1 + v_2 + v_3 + \text{--------}$

are two convergent series they may be added; the resulting series is convergent and has for its sum $U + V$ or

$$U + V = u_1 + v_1 + u_2 + v_2 + \text{---------}.$$

Let

$$s_n = u_1 + u_2 + u_3 + \text{----------} + u_n$$

$$t_n = v_1 + v_2 + v_3 - \text{----------} + v_n$$

$$S_n = (u_1 + v_1) + (u_2 + v_2) + \text{---} + (u_n + v_n)$$

To show that $\lim\limits_{n \to \infty} S_n = U + V$

Now since $S_n = \sum (u_n + v_n) = \sum u_n + \sum v_n = s_n + t_n$

we have only to pass to the limit as n approaches infinity and we have

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n$$

or

$$\lim_{n \to \infty} S_n = U + V$$

If two variables are always equal while approaching their respective limits their limits are equal.

Should the last parentheses be incomplete and there should be one u_n or a v_n left uncombined, this could not affect the value of series for large values of n because the given series are convergent and the $\lim\limits_{n \to \infty} u_n = 0$ and $\lim\limits_{n \to \infty} v_n = 0$.

Corollary 1: A finite number of convergent series may be added by applying Theorem II, case 1, n times in succession.

Multiplication of Series

Theorem 6: Convergent series may be multiplied by a constant, that is to say if $\sum u_n = S$ it follows if c is any arbitrary number that $\sum cu_n = cS$.

The partial sums of the new series are cs_n if those of the old are s_n

$$\lim_{n\to\infty} \sum cu_n = \lim_{n\to\infty} cs_n = cS$$

In the multiplication of one polynomial by another, we multiply each term of the one by every term of the other, and take the sum of the products formed. For example:

$$S_n = u_1 + u_2 + --- + u_n$$

$$T_n = v_1 + v_2 + --- + v_n$$

The product of the polynomials may be written S_nT_n =

$$\begin{array}{ccccc}
+ u_1v_1 & + u_1v_2 & + u_1v_3 & + \text{----} & + u_1v_n \\
+ u_2v_1 & + u_2v_2 & + u_2v_3 & + \text{----} & + u_2v_n \\
+ u_3v_1 & + u_3v_2 & + u_3v_3 & + \text{----} & + u_3v_n \\
\vdots & \vdots & \vdots & & \vdots \\
u_nv_1 & + u_nv_2 & + u_nv_3 & + \text{---} & + u_nv_n
\end{array}$$

If we allow n to become infinite, we are led to the product of two infinite series. It is convenient to exhibit such a

product by grouping the terms in the above array by diagonals.

For example: the product of the series is

$$u_1v_1 + u_2v_1 + u_1v_2 + u_3v_1 + u_2v_2 + u_1v_3 + \text{----------------}$$

$$+ u_nv_1 + u_{n-1}\ v_2 + \text{----} + u_1v_n .$$ The question is

then, under what condition does this series converge to U V: that is, under what conditions do we have the following:

$$\lim_{n \to \infty} s_n \cdot t_n = \lim_{n \to \infty} s_n \cdot \lim_{n \to \infty} t_n = U\ V$$

Theorem 7:

If

$$U = u_1 + u_2 + u_3 + \text{----}$$

$$V = v_1 + v_2 + v_3 + \text{----}$$

are any two absolutely convergent series, they can be multiplied together like sums, that is, if each term in the first series be multiplied into every term in the second and the series of these products formed, this series will converge absolutely toward the limit UV.[1] For example:

$$UV = u_1v_1 + u_1v_2 + u_2v_1 + u_1v_3 + u_2v_2 + u_3v_1 + \text{---------}$$

Let

$$s_n = u_1 + u_2 + u_3 + \text{---} + u_n$$

$$t_n = v_1 + v_2 + v_3 + \text{----} + v_n$$

then $\lim s_nt_n = UV$.

The terms in s_nt_n may be arranged in the array as above in a

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1. Osgood, William F., Introduction to Infinite Series, p. 46.

square, n terms on a side and we have

$$
\begin{array}{l}
u_1v_1 + u_1v_2 + u_1v_3 + u_1v_4 + ---- \\
u_2v_1 + u_2v_2 + u_2v_3 + u_2v_4 + ---- \\
u_3v_1 + u_3v_2 + u_3v_3 + u_3v_4 - ---- \\

\end{array}
$$

The theorem asserts that if any series formed by adding the terms of this scheme, each term appearing in this series once and only once, for example, the terms that lie on the oblique lines, the successive lines being followed from top to bottom:

(a) $\quad u_1v_1 + u_1v_2 + u_2v_1 + u_1v_3 + u_2v_2 + u_3v_1 + -----,$

this series will converge absolutely to the limit UV.

It is sufficient to show that one series formed in the prescribed way form the terms in this scheme, for example, the series formed by following the successive boundaries of the squares from top to bottom and then from right to left, namely the series

(b) $\quad u_1v_1 + u_1v_2 + u_2v_2 + u_2v_1 + u_1v_3 + -------$

converges absolutely toward limit UV. For any other series may be generated by a rearrangement of the terms in this series.

Let S_N denote the sum of the first N terms in (b).

First, suppose all the terms in the u series and v series to be positive

Then if $\quad n^2 \leqq N < (n+1)^2$

$$s_n t_n \leqq S_N \leqq s_{n+1} \cdot t_{n+1}$$

But $\quad \lim\limits_{n\to\infty} s_n t_n = \operatorname{Lim}\limits_{n\to\infty} s_{n+1} \cdot t_{n+1} = UV$

$$\therefore \quad \lim_{N\to\infty} S_N = UV$$

Secondly, if the u series and the v series are any absolutely convergent series, form the series of absolute values.

$$|u_1| + |u_2| + |u_3| + ----$$

$$|v_1| + |v_2| + |v_3| + ---$$

The product of these series is the convergent series

(c) $$|u_1|\,|v_1| + |u_1|\,|v_2| + |u_2|\,|v_1| + |u_1|\,|v_3| + -----$$

But this series is precisely the series of absolute values of (b) and therefore converges absolutely. It remains to show that it converges toward value UV.

Since S_N approaches a limit when N, increasing, passes through all integral values, since the series (c) is absolutely convergent, S_N will continue to approach a limit, and this will be the same limit, if N passes through only the values n^2.

$$\lim_{N\to\infty} S_N = \lim_{n\to\infty} s_{n^2}$$

But $$s_{n^2} = s_n t_n \qquad \text{and} \lim_{n\to\infty} S_{n^2} = UV$$

Division of Series

Theorem 8: If the two series

$$U = u_1 + u_2 + u_3 + u_4 + \text{------}$$

$$V = v_1 + v_2 + v_3 + v_4 + \text{------}$$

are convergent and the v series is absolutely convergent with $v_1 \neq 0$ $V \neq 0$

then

$$\frac{\sum u_n}{\sum v_n} = \frac{U}{V} = \sum x_n$$

where $x_n = \dfrac{u_n - x_1 v_n - x_2 v_{n-1} - \text{--------} x_{n-1} V_2}{v_1}$

provided that $\sum x_n$ is absolutely convergent.[1]

For in order that we may have

$\dfrac{\sum u_n}{\sum v_n} = \sum x_n$, the series $\sum u_n$ must be the product of

$\sum v_n$ and $\sum x_n$.

Since both $\sum v_n$ and $\sum x_n$ converge absolutely by hypothesis, they may be multiplied (two absolutely convergent series may be multiplied and the product is an absolutely convergent series), and the product be written

$$x_1v_1 + (x_1v_2 + x_2v_1) + (x_1v_3 + x_2v_2 + x_3v_1) +$$

$$(x_1v_n + x_2v_n + \text{-------} + x_nv_1) + \text{----}$$

If this series is to be identical with the u series, the

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1. Smail, Lloyd L., Elements of the Theory of Infinite Processes, p. 124.

values x_1 , x_2 , x_3 , must be determined so that the corresponding terms are equal. Then

$$u_1 = x_1v_1; \quad u_2 = (x_1v_2 + x_2v_1); \quad u_n = (x_1v_2 + x_2v_{n-1} + \cdots + x_nv_1).$$

Solving for the x's since $v_1 \neq 0$, we get $x_1 = \dfrac{u_1}{v_1}$:

$$x_2 = \frac{u_2 - x_1v_2}{v_1}; \qquad x_3 = \frac{u_3 - x_1v_3 - x_2v_2}{v_1}$$

and $$x_n = \frac{u_n - x_1v_n - x_2v_{n-1} - \cdots - x_{n-1}\ v_2}{v_1}$$

Example of Division

$$\frac{u_n}{v_n} = \frac{1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} - \cdots}{\frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} - \cdots}$$

By substituting the values of u_n and v_n in the formula for x_n we have

$$\sum x_n = 2 - \frac{1}{6} - \frac{11}{54} - \frac{5003}{51840} - \cdots$$

By long division we get the same series:

$$\frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \cdots \quad \Big) \quad 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^6} + \cdots \qquad \text{quotient: } 2 - \frac{1}{6} - \frac{11}{54} - \frac{5003}{51840} - \cdots$$

$$1 + \frac{1}{3} + \frac{1}{2.3} + \frac{1}{2.5} + \frac{1}{15} + \cdots$$

$$- \frac{1}{12} - \frac{7}{54} - \frac{123}{1280} - \frac{622}{9375} - \cdots$$

$$- \frac{1}{12} - \frac{1}{36} - \frac{1}{72} - \frac{1}{120} - \cdots$$

$$- \frac{11}{108} - \frac{947}{11520} - \frac{4351}{75000} - \cdots$$

$$- \frac{11}{108} - \frac{11}{324} - \frac{11}{648} - \cdots$$

$$- \frac{5003}{103680} - \cdots$$

<u>Conclusion</u>, Part II.

We see, then, that for absolutely convergent series, the fundamental laws of algebra are in all essentials maintained, but they do not hold for infinite series in general.

The sum of an infinite number of terms in an absolutely convergent series remains unaltered, however the terms may be arranged, so that the commutative law of algebra holds unrestrictedly.

The sum of any number of convergent series gives a series, convergent and equal to the sum of the series.

Two series may be multiplied term by term as algebraic polynomials, provided that they are absolutely convergent, and the product is a convergent series, also a finite number of absolutely convergent series multiplied and the product is absolutely convergent series.

Absolutely convergent series may be divided one by the other and the quotient is absolutely convergent, provided that the first term in the divisor is not zero $(v_1 \neq 0)$ and $V \neq 0$.

Part III

Power Series

A. Definition.

By a power series we shall understand a series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \text{-----} + a_nx^n + \text{----------}$$

where n is a positive integer, and the a's are constants. If n is negative or fractional, the series may be spoken of as a power series with negative or fractional exponents, as the case may be. Properties of power series are shown in the following theorems:

B. Theorems of power series whose proofs are assumed.

Fundamental Theorem on Power series.

Theorem 1: For a power series $\sum a_nx^n$ which converges, for a value of $x \neq 0$ but is not always convergent, there exists a positive number R such that for $|x| < R$, the series is convergent, and for $|x| > R$, the series is divergent.

Definition of radius of convergence:

Suppose we have a positive value R which satisfies the condition $\sum a_nx^n$ converges for $|x| < R$ and diverges for $|x| > R$. The interval (-R, R) is then called the interval of convergence of the given power series, and it is often convenient to speak of R as the radius of convergence. Thus the power series $\sum_{n=0}^{\infty} x^n$ has unity as its radius of convergence and its interval of convergence is (-1, 1). The interval may be an open or closed interval, depending on the nature of the series when $x = \pm 1$.

Theorem 2: The binomial series $(1 + x)^m$ is absolutely convergent for $|x| < 1$ for all values of m.

Theorem 3: A power series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \text{-------}$$

represents a continuous function f(x) within the interval of convergence.

C. Theorems to be Proved:

Theorem 1: Two power series may be added term by term and the resulting series will be convergent, for values of x in the interior of both of the intervals of convergence of the given series.

Let

$$A = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \text{-------}$$

be a power series convergent for $|x| < R$.

Let

$$B = b_0 + b_1x + b_2x^2 + b_3x^3 + \text{----}$$

be a power series convergent for $|x| < R'$.

Let x_0 be any value of x such that $R > |x_0| < R'$.

Then

$$A = a_0 + a_1x_0 + a_2x_0^2 + a_3x_0^3 + \text{------}$$

and

$$B = b_0 + b_1x_0 + b_2x_0^2 + b_3x_0^3 + \text{-----}$$

converge absolutely, for x_0 is less than the radius of convergence of both series.

Hence

$$A + B = (a_o + b_o) + (a_1 + b_1)\, x_o + (a_2 + b_2)\, x_o^2 + (a_3 - b_3)\, x_o^3 + \text{-----}.$$

For two absolutely convergence series may be added (Part II, 5) and their sum is a convergent series.

Example: $$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \text{-----------} \quad -1 < x < 1$$

$$1 + \frac{x}{2} + \frac{x^2}{2.2^2} + \frac{x^3}{3.2^3} + \text{---------} \quad -2 < x < 2 .$$

Then by adding,

$$1 + \frac{3x}{2} + \frac{5x^2}{2.2^2} + \frac{9x^3}{3.2^3} + \frac{17x^4}{4.2^4} + \text{-----} \qquad -1 < x < 1$$

Theorem 2: Power series may be multiplied term by term and the product is a convergent series, provided that x is in the interior of the intervals of convergence of both the given series.

Let

$$a_o + a_1 x + a_2 x^2 + a_3 x^3 + \text{-----}$$

be a power series convergent for $|x| < R$

and let

$$b_o + b_1 x + b_2 x^2 + b_3 x^3 + \text{----}$$

be a power series convergent for $|x| < R'$

Let x_o be any x such that $R > |x_o| < R'$

Then $$a_o + a_1 x_o + a_2 x^2 + a_3 x^3 + \text{----}$$

and $$b_o + b_1 x_o + b_2 x_o^2 + b_3 x_o^3 + \text{----}$$

will converge absolutely since x_o is less than the radius of convergence of both series.

Hence

$$a_ob_o + (a_1b_o + a_ob_1)\ x_o + (a_2b_o + a_ob_2 + a_1b_1)\ x_o^2 + ----$$

is an absolutely convergent series, for two absolutely convergent series may be multiplied and their product is a convergent series (Part II, 7).

Corollary I. A finite number of power series may be multiplied together by applying theorem 2 successively.

Corollary II. A power series may be multiplied by itself an arbitrary number of times and the product will be convergent within the interval of convergence of the given series.

Example of Multiplication of series:

$$\sin x = \cos x \tan x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \text{------}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \text{------}$$

$$x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \frac{x^9}{8!} \text{------------------}$$

$$+ \frac{x^3}{3} - \frac{x^5}{3.2!} + \frac{x^7}{3.4!} - \frac{x^9}{3.6!} - \text{-------------}$$

$$\frac{2x^5}{15} - \frac{2x^7}{15.2!} + \frac{2x^9}{15.4!} \text{---------}$$

$$+ \frac{17x^7}{315} - \frac{17x^9}{315.2!} - \text{---------}$$

$$\frac{62x^9}{2835} - \text{---------}$$

$$\text{---------}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \text{--------------------}.$$

convergent for all values of x.

Power series of power series.

Theorem 3: If the power series[1]

$$a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \text{-----}$$

is convergent for $|y| < R$ and if the power series

$$y = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \text{-----}$$

is convergent for $|x| < r$, the y series may be developed into a power

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1. Wilson, Edwin B., Advanced Calculus, p. 445.

series in x, convergent about the point $x = 0$ provided that $|b_0| < R$.

Since a power series represents a continuous function in the interval of convergence (Part 3 B 3) and $|b_0| < R$, $|b_0|$ is within the interval of convergence of

$$a_0 + a_1 y + a_2 y^2 + a_3 y^3 + ----$$

If $y = f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + -----$

when $x = 0$ $y = b_0$. For sufficiently small values of x, y will lie within the interval of convergence R about b_0. Hence the y series converges.

To determine the coefficients, by the theorem on multiplication y may be squared, cubed, etc.

$$y^2 = b_0^2 + 2b_0 b_1 x + (b_1^2 + 2b_0 b_2)\, x^2 + (2b_1 b_2 + 2b_0 b_3)\, x^3 - ---$$

$$y^3 = b^3 + 3b_0^2 b_1 x + 3(b_0 b_1^2 + b_0^2 b_2)\, x^2 + -----$$

Substituting these values in

$$a_0 + a_1 y + a_2 y^2 + a_3 y^3 + ----------------------$$

and arranging terms in the powers of x we have

$$a + a_1 b_0 + a_1 b_1 x + a_1 b_2 x^2 + a_1 b_3 x^3 + ---$$

$$+ a_2 b_0^2 + 2a_2 b_0 b_1 x + a_2 (b_1^2 + 2b_0 b_2) x^2 - -----$$

$$+ a_3 b_0^3 + 3a_3 b_0^2 b_1 x + 3a_3 (b_0 b_1^2 + b_0^2 b\ ) x^2$$

----

or

$$(a_0 + a_1 b_0 + a_2 b_0^2 + a_3 b_0^3 + ---) + (a_1 b_1 + 2a_2 b_0 b_1 + 3a_3 b_0^2 b_1 + --)x$$

$$+ \left[a_1 b_2 + a_2 (b_1^2 + 2b_0 b_2) + 3a_3 (b_0 b_1^2 + b_0^2 b_2) + - ----\right] x^2 + ----$$

Consider the problem of expanding $e^{\sin x}$ to five terms.

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^5}{5!} + \cdots\cdots$$

where $y = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots\cdots$

$$y^2 = x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 - \cdots\cdots$$

$$y^3 = x^3 - \frac{1}{2}x^5 + \frac{13}{120}x^7 - \cdots$$

$$y^4 = x^4 - \frac{5}{6}x^6 \cdots\cdots\cdots\cdots$$

$$y^5 = x^5 - \cdots\cdots\cdots\cdots$$

$$\begin{array}{llllll}
e^y = 1 + x & & -\frac{x^3}{6} & & +\frac{x^5}{120} & \cdots\cdots\cdots \\
& +\frac{1}{2}x^2 & & -\frac{1}{6}x^4 & & +\frac{1}{45}x^6 \cdots \\
& & +\frac{x^3}{6} & & -\frac{1x^5}{12} - \cdots & \\
& & & +\frac{x^4}{24} + \frac{1}{120}x^5 & & -\frac{5}{36}x^6 - \cdots
\end{array}$$

Summing by columns

$$e^y = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{15}x^5 + \cdots\cdots\cdots\cdots$$

convergent for all values of x.

Theorem 4: The quotients of two convergent power series

$$\frac{b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots\cdots \qquad |x| < r}{a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\cdots \qquad |x| < R}$$

may be expressed as a power series convergent in the neighborhood of $x = 0$.

$$c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots\cdots \qquad |x| < r \quad |x| < R.$$

provided that $a_0 \neq 0$

[illegible]

(a) [illegible]

(b) [illegible]

[illegible]

It is sufficient to show that

$$\frac{1}{a_0 + a_1x + a_2x^2 + a_3x^3 + \text{-----}}$$

can be expressed as a power series then this power series may be multiplied by

$$b_0 + b_1x + b_2x^2 + b_3x^3 + \text{---------}$$

Divide the power series $a_0 + a_1x + a_2x^2 + \text{---}$

by a_0 and we have

$$1 + \frac{a_1}{a_0}x + \frac{a_2}{a_0}x^2 + \frac{a_3}{a_0}x^3 + \text{-----}$$

Let $\dfrac{a_n}{a_0} = -a'_n$

where n = 1, 2, 3, ----

Then the series becomes

$$1 - (a'_1x + a'_2x^2 + a'_3x^3 + a'_4x^4 + \text{-----------})$$

and put

$$y = (a'_1x + a'_2x^2 + a'_3x^3 + \text{--------})$$

We have then

$$\frac{1}{a_0 + a_1x + a_2x^2 + \text{----}} = \frac{1}{a_0} \cdot \frac{1}{1-y}$$

Now $\dfrac{1}{1-y}$ or $(1-y)^{-1}$ is a geometric and also a binomial series

$$1 + y + y^2 + y^3 + \text{-------}$$

which is convergent for $|y| < 1$. (Part III B Theorem 2)

Since y approaches zero as x approaches zero, $|y| < 1$ for values of x in the neighborhood of zero, the a' series converges for small values

of x.

We have then

$$\frac{1}{a_0 + a_1x + a_2x^2 + ---} = \frac{1}{a_0}\ (1 + y + y^2 + y^3 + -----) =$$

$$\frac{1}{a_0}\Big[1 + (a_1' x + a_2' x^2 + a_3' x^3 + ---) + (a_1' x + a_a' x^2 + -----)^2 +$$

$$(a_1' x + a_2' x^2 + a_3' x^3 + ------)^3 + ------------.\Big]$$

by substituting a power series in a power series (Part III C Theorem 3). By expanding the parentheses and collecting the terms containing the like powers of x we have a power series convergent about the point $x = 0$. Multiply this series by

$$b_0 + b_1x + b_2x^2 + b_3 x^3 + --------$$

and we have the power series

$$c_0 + c_1x + c_2x^2 + c_3x^3 + ----------$$

convergent in the neighborhood of $x = 0$. for the product of two convergent power series gives a convergent power series.

Example of Division.

$$\frac{\sum b_n x^n}{\sum a_n x^n} = \frac{1 + 2x + 3x^2 + 4x^3 + 5x^4 + \text{---}}{1 - 3x + 5x^2 - 7x^3 + 9x^4 - \text{---}} \qquad \begin{array}{l} -1<x<1 \\ -1<x<1 \end{array}$$

$a_0 \neq 0$

$$\begin{array}{rl|l}
1 + 2x + 3x^2 + 4x^3 + 5x^4 + \text{---} & & 1 - 3x + 5x^2 - 7x^3 + 9x^4 - \text{---} \\
\underline{1 - 3x + 5x^2 - 7x^3 + 9x^4 - \text{---}} & & \underline{1 + 5x + 13x^2 + 25x^3 + 41x^4 + \text{-----}} \\
5x - 2x^2 + 11x^3 - 4x^4 + \text{---} & & \\
\underline{5x - 15x^2 + 25x^3 - 35x^4 + \text{---}} & & \\
13x^2 - 14x^3 + 31x^4 - \text{---} & & \\
\underline{13x^2 - 39x^3 + 65x^4 - \text{---}} & & \\
25x^3 - 34x^4 + \text{---} & & \\
\underline{25x^3 - 75x^4 + \text{---}} & & \\
+ 41x^4 - \text{---} & &
\end{array}$$

Theorem 5: If two power series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \text{-------}$$

and

$$b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \text{-------}$$

have the same value for all values of x in the neighborhood of $x = 0$ the corresponding coefficients in the two series must be identical.

If we write

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \text{----} - (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \text{------}) = 0$$

we have

$$(a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)\,x^2 + \text{-------} = 0$$

Now when $x = 0$ $a_0 - b_0 = 0$ or $a_0 = b_0$

Write the reduced series in the form

$$x\left[(a_1 - b_1) + (a_2 - b_2)x + (a_3 - b_3)x^2 + ---- \right] = 0$$

Now if we put

$$f(x) = (a_1 - b_1) + (a_2 - b_2)x + ---- + (a_n - b_n)x^{n-1} + ---$$

Since a power series represents a continuous function within the interval of convergence

$$\lim_{x\to 0} f(x) = f(0) = a_1 - b_1$$

But, $\lim_{x\to 0} f(x) = 0$

Therefore $a_1 - b_1 = 0$ $a_1 = b_1$

By repeating the process

$$a_n - b_n = 0 \quad \text{or} \quad a_n = b_n$$

Reversion of Series

Theorem: If the series[1]

(1) $u = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots\cdots\cdots \qquad b_1 \neq 0$

converges when $|x| \leqq x_0$, $x_0 > 0$ there exists an $h > 0$ such that when $(u-b_0) < h$ there is a unique series of the type

(2) $x = (u-b_0)\left\{\frac{1}{b_1} + a_1(u-b_0) + a_2(u-b_0)^2 + \cdots\cdots\cdots\right\}$

which converges and satisfies (1).

Proof: Replace x by tx_0 and let $y = \frac{u-b_0}{b_1x_0}$

then series (1) reduces to the form

(3) $y = t - c_2t^2 - c_3t^3 - c_4t^4 - \cdots\cdots\cdots.$

which converges for $t = 1$ ($t = 1$, $x = x_0$), and series (2) reduces to a series of the form.

(4) $t = y(1 + d_1y + d_2y^2 + d_3y^3 + \cdots\cdots\cdots.)$

If t is given by (4) the corresponding value of x is given by (2) since $tx_0 = x$.

There consequently is no loss of generality in treating the latter series and we shall proceed to do so.

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1. Fort, Tomlinson, Infinite Series, p. 142.

Substitute (4) in (3)

$$t = d_1y + d_2y^2 + d_3y^3 + d_4y^4 + \text{---------}.$$

$$-c_2t^2 = -c_2y^2 - 2c_2d_2y^3 - c_2(d_2^2 + 2d_3)y^4 - c_2(2d_2d_3 + 2d_4)y^5 - \text{---}$$

$$-c_3t^3 = -c_3y^3 - 3c_3d_2y^4 - c_3(3d_2^2 + 3d_3)y^5 - \text{-------}.$$

$$-c_4t^4 = -c_4y^4 - 4c_4d_2y^5 - \text{------}.$$

$$-c_5t^5 = -c_5y^5 - \text{-------}$$

We get

$$y = d_1y + (d_2 - c_2)y^2 + (d_3 - 2c_2d_2 - c_3)y^3$$

$$+ \left[d_4 - c_2(d_2^2 + 2d_3) - 3c_3d_2 - c_4\right] y^4$$

$$+ \left[d_5 - c_2(2d_2^2d_3 + 2d_4) - c_3(3d_2^2 + 3d_3) - 4c_4d_2 - c_5\right] y^5 + \text{-----}$$

Equate the coefficients of like powers of y we have by Part III C Theorem 5

$$d_1 = 1$$

$$d_2 = c_2$$

$$d_3 = 2c_2d_2 + c_3$$

$$d_4 = c_2(d_2^2 + 2d_3) + 3c_3d_2 + c_4$$

$$d_5 = 2c_2(d_2d_3 + d_4) + 3c_3(d_2^2 + d_3) + 4c_4d_2 + c_5$$

$$d_n = f_n\,(c_1 \text{ ------ } c_{n-1}) + c_n$$

f_n being a polynomial in the c's with positive coefficients. We see that d_n's are always uniquely determinable since the c's are known. This determination gives a formal solution to the problem. It remains

then to prove the convergence of (4) for values other than zero. Series (3) converges when $t = 1$. Hence $|c_n| < \alpha$ a constant since when $t = 1$ the series is convergent, hence the $\lim_{n\to\infty} c_n = 0$.

(5) $|d_n| \leqq f_n(\alpha ---- \alpha) + \alpha = F_n \alpha$.

Now let us consider (3) substituting α for c_n we have

(6) $y = t - \alpha t^2 - \alpha t^3 - --------.$

which becomes a geometric series with sum

$$y = t - \frac{\alpha t^2}{1-t}$$

Solving for t

$$y - yt = t - t^2 - \alpha t^2 \text{ or } (\alpha + 1)t^2 - t(y + 1) - y = 0$$

(7) we find $t = \dfrac{y + 1 \pm \sqrt{1-2(2\alpha + 1)y + y^2}}{2(\alpha + 1)}$

Write the radical thus determined as

$$(1-2(2\alpha + 1)y - y^2)^{\frac{1}{2}}$$

and expand by the binomial theorem getting a power series in y which converges when $|2(2\alpha + 1)y - y^2| < 1$

Substitute this series in (7) using the minus sign before the radical which causes the constant terms in the numerator to vanish leaving t equal to a series in y, so that when $y = 0$, $t = 0$ and the series will converge for values of y near zero.

If we substitute

$$t = d_1 y + d_2 y^2 + d_3 y^3 + d_4 y^4 + ---------$$

in $y = t - \alpha t - \alpha t^2 - \alpha t^3$ ------------

as we substituted (4) in (3) we could uniquely determine the coefficients of this series as we did then. This series would have to coincide with the series derived from (7) as they represent the inverse of the same series. We have shown that the series derived from (7) converges for values of y near zero, so the series derived by the substitution must converge within the same limits.

Since $|d_n| \leq F_n \alpha$ series (4) must converge for small values of y because all its coefficients are less than or equal to those in the series derived by the substitution.

Note: If $b_0 = 0$, x becomes a series in u.

Example of reversion:

I. Given $y = 2x + 3x^2 - x^3 + 2x^4 + 3x^5 - x^6 + \text{------}$

To find the series

$$x = b_1y + b_2y^2 + b_3y^3 + \text{--------}$$

$$b_1 = \tfrac{1}{2}$$

$$x = b_1(2x + 3x^2 - x^3 + 2x^4 + \text{-----})$$

$$- b_2(2x + 3x^2 - x^3 + 2x^4 + \text{----})^2$$

$$- b_3(2x + 3x^2 - x^3 + 2x^4 + \text{-----})^3$$

$$- b_4(2x + 3x^2 - x^3 + 2x^4 + \text{-----})^4$$

$$x = +2b_1 \left| \; x \quad \begin{matrix} +3b_1 \\ +4b_2 \end{matrix} \right| x^2 \quad \begin{matrix} -b_1 \\ +12b_2 \\ +8b_3 \end{matrix} \left| x^3 \quad \begin{matrix} +2b_1 \\ +5b_2 \\ +32b_3 \\ +16b_4 \end{matrix} \right| x^4 \; + \; \begin{matrix} 3b_1 \\ +2b_2 \\ +42b_3 \\ +96b_4 \\ +32b_5 \end{matrix} \Big| x^5$$

$2b_1 = 1$ $\qquad 3b_1 + 4b_2 = 0$ $\qquad -b_1 + 12b_2 + 8b_3 = 0$

$b_1 = \frac{1}{2}$ $\qquad \frac{3}{2} = -4b_2$ $\qquad -\frac{1}{2} - \frac{9}{2} = -8b_3$

$-\frac{3}{8} = b_2$ $\qquad 5 = 8b_3$

$2b_1 + 5b_2 + 36b_3 + 16b_4 = 0$ $\qquad \frac{5}{8} = b_3$

$3b_1 + 2b_2 + 42b_3 + 96b_4 + 32b_5 = 0$

$1 - \frac{15}{8} + \frac{180}{8} = -16b_4$ $\qquad \frac{12}{8} - \frac{6}{8} + \frac{210}{8} - \frac{1038}{8} = -32b_5$

$+\frac{173}{8} = -16b_4$ $\qquad -\frac{822}{8} = -32b_5$

$-\frac{173}{128} = b_4$ $\qquad +\frac{411}{128} = b_5$

$$x = \frac{1\cancel{x}}{2} - \frac{3\cancel{x}^2}{8} + \frac{5\cancel{x}^3}{8} - \frac{173\cancel{x}^4}{128} + \frac{411x^5}{128} - \text{-------}$$

CONCLUSION

In dealing with power series the fundamental laws of algebra are maintained when the series is considered within the interval of convergence for within this interval the power series represents a continuous function.

The theorems of addition, multiplication and division which apply to absolutely convergent series, with certain restrictions as to coefficients and constant terms apply to powers series, for a power series is absolutely convergent, for all values of the unknown in the interval of convergence. The results will be convergent for values of the unknown which are in the interior of the intervals of convergence of the series considered.

Thus we have seen infinite series transformed by the simple algebraic processes and have determined what restrictions are necessary for the application of these laws.

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